Indefinite Integral

Let $f(x)$ be a function, the family of all its primitives (or antiderivatives) is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)dx$

1. Standard Intergrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$ (ii) $\int \frac{1}{x} dx = \log|x| + C$ (iii) $\int e^x dx = e^x + C$ (iv) $\int a^x dx = \frac{a^x}{\log a} + C$ (v) $\int \sin x dx = -\cos x + C$ (vi) $\int \cos x dx = \sin x + C$

(vii) $\int \sec^2 x dx = \tan x + C$ (viii) $\int \operatorname{cosec}^2 x dx = -\cot x + C$ (ix) $\int \sec x \tan x dx = \sec x + C$ (x) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$ (xi) $\int \cot x dx = \log|\sin x| + C$

(xii) $\int \tan x dx = \log|\sec x| + C$ (xiv) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$ (xv) $\int \sec x dx = \log|\sec x + \tan x| + C$ (xvi) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$ (xviii) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ (xx) $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$ (xix) $\int -\frac{1}{a^2 + x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(xxi) $\int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

2. Integration By Substitution

Expression	Substitution
$a^2 + x^2$	$x = a\tan\theta$ or $a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta - x}}$ or, $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

3. Integration Using Partial Fractions

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$ (ii) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorised further.

4. Integration By Parts

$\int u \cdot v dx = u\int v dx - \int\left[\frac{du}{dx}\cdot\int v dx\right]dx$

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5. QUIK LOOK

- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$
- $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$
- $\int\left[xf'(x) + f(x)\right]dx = xf(x) + C$
- $\int\sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int\sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int\sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

6. Integrals of different forms:

(1) $\int \sin^m x dx, \int \cos^m x dx$, where $m \leq 4$

express $\sin^m x$ and $\cos^m x$ in terms of sines and cosines of multiples of x by using the following identities:

(i) $\sin^2 x = \frac{1 - \cos 2x}{2}$ (ii) $\cos^2 x = \frac{1 + \cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$ (iv) $\cos 3x = 4\cos^3 x - 3\cos x$

(2) $\int \sin mx \cos nx dx, \int \sin mx \sin nx dx, \int \cos mx \cos nx dx$

use the following trigonometrical identities:

$2\sin A\cos B = \sin(A+B) + \sin(A-B); 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B); 2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(3) $\int \frac{f'(x)}{f(x)}dx = \log\{f(x)\} + C$ (4) $\int \{f(x)\}^n f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1}, n \neq -1$

(5) $\int \tan^m x \sec^{2n} x dx, \int \cot^m x \operatorname{cosec}^{2n} x dx; m, n \in N$

Put $\tan x = t$ and $\sec^2 x dx = dt$

(6) $\int \sin^m x \cos^n x dx, m, n \in N$

If the exponent of sin x is an odd positive integer put cos$x = t$

If the exponent of cos x is an odd positive integer put sin$x = t$.

(7) $\int \sin^m x \cos^n x dx$, Where $m, n \in Q$, $m+n$ is a negative even integer

Change the integrand in terms of $\tan x$ and $\sec^2 x$ by dividing numerator and denominator by $\cos^k x$, where $k=-(m+n)$ then put $\tan x = t$

(8) $\int \frac{1}{ax^2+bx+c}dx$

express ax^2+bx+c as the sum or difference of two squares.

(9) $\int \frac{px+q}{ax^2+bx+c}dx$

To evaluate this,

$$px+q=\lambda\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+\mu \text{ i.e. } px+q=\lambda(2ax+b)+\mu$$

(10) $\int (px+q)\sqrt{ax^2+bx+c}dx$

In order to evaluate this, write

$$px+q=\lambda\frac{d}{dx}\left(ax^2+bx+c\right)+\mu \text{ i.e. } px+q=\lambda(2ax+b)+\mu$$

(11) $\int \frac{P(x)}{ax^2+bx+c}dx$, where p (x) is a polynomial of degree two or more

to evaluate this, write $\int \frac{P(x)}{ax^2+bx+c}dx=\int Q(x)dx+\int \frac{R(x)}{ax^2+bx+c}dx$

(12) $\int \frac{1}{a\sin^2 x+b\cos^2 x}dx, \int \frac{1}{a+b\sin^2 x}dx$

$\int \frac{1}{a+b\cos^2 x}dx, \int \frac{1}{(a\sin x+b\cos x)^2}dx, \int \frac{1}{a+b\sin^2 x+c\cos^2 x}dx$

To evaluate this type of integrals, divide numerator and denominator both by $\cos^2 x$

(13) $\int \frac{1}{a\sin x+b\cos x}dx, \int \frac{1}{a+b\sin x}dx, \int \frac{1}{a+b\cos x}dx$

$\int \frac{1}{a\sin x+b\cos x+c}dx$

To evaluate this, put $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}$ and, $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

and simplify.

(14) $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$

To evaluate this, write Numerator
$=\lambda$(Diff. of denominator)) $+\mu$(Denominator))

(15) $\int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

(16) $\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$

(17) $\int (px+q)\sqrt{ax^2+bx+c}dx$

In order to evaluate this, write

$$px+q=\lambda\frac{d}{dx}\left(ax^2+bx+c\right)+\mu \text{ i.e. } px+q=\lambda(2ax+b)+\mu$$

(18) $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$

put $cx+d=t^2$

(19) $\int \frac{1}{\left(ax^2+bx+c\right)\sqrt{px+q}}dx$

put $px+q=t^2$

(20) $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}dx$

put $ax+b=\frac{1}{t}$

(21) $\int \frac{1}{\left(ax^2+b\right)\sqrt{cx^2+d}}dx$

put $x=\frac{1}{t}$ to obtain

(22) $\int \frac{-tdt}{\left(a+bt^2\right)\sqrt{c+dt^2}}$

substitute $c+dt^2=u^2$

Reduction formulas

7.

(1) Reduction Formula for Exponential Functions

- $\int x^n e^{mx}dx=\left[(1/m)x^n e^{mx}\right]-\left[(n/m)\int x^{n-1}e^{mx}\right]dx$
- $\int e^{mx}/x^n dx=-\left[e^{mx}/(n-1)x^{n-1}\right]+\left[(m/n-1)\int e^{mx}/x^{n-1}\right]dx, n\neq 1$

(2) Reduction Formula for Trigonometric Functions

- $\int \sin^n(x)dx=\frac{-\mathrm{Sin}^{n-1}(x)\mathrm{Cos}(x)}{n}+\frac{n-1}{n}\int \mathrm{Sin}^{n-2}(x)dx$
- $\int x^n\cos(x)dx=x^n\sin(x)-n\int x^{n-1}\sin(x)dx$
- $\int x^n\sin(x)dx=-x^n\cos(x)+n\int x^{n-1}\cos(x)dx$
- $\int \sin^n(x)\cos^m(x)dx=\frac{\sin^{n+1}(x)\cos^{m-1}(x)}{n+m}+\frac{m-1}{n+m}\int \sin^n(x)\cos^{m-2}(x)dx$
- $\int \frac{dx}{\sin^n x}=-\frac{\cos x}{(n-1)sin^{n-1}x}+\frac{(n-2)}{(n-1)}\int \frac{dx}{\sin^{n-2}x}, n\neq 1$
- $\int \frac{dx}{\cos^n x}=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{(n-2)}{(n-1)}\int \frac{dx}{\cos^{n-2}x}, n\neq 1$
- $\int \tan^n(x)dx=\frac{\tan^{n-1}(x)}{n-1}-\int \tan^{n-2}(x)dx$

(3) Reduction Formula for Logarithmic Functions

- $\int x^n \ln^m x dx=\frac{x^{n+1}ln^m x}{n+1}-\frac{m}{n+1}\int x^n \ln^{m-1}xdx$
- $\int \frac{ln^m x}{x^n}=-\frac{ln^m x}{(n-1)x^{n+1}}+\frac{m}{n-1}\int \frac{l^{m-1}x}{x^n}dx, n\neq 1$

(4) Reduction Formula for Inverse Trigonometric Functions

- $\int x^n \arcsin xdx=\frac{x^{n+1}}{n+1}\arcsin x-\frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$
- $\int x^n \arccos xdx=\frac{x^{n+1}}{n+1}\arccos x+\frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$
- $\int x^n \arctan xdx=\frac{x^{n+1}}{n+1}\arctan x-\frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1+x^2}}dx$

(5) Reduction Formula for Algebraic Functions

- $\int \frac{dx}{\left(ax^2+bx+c\right)^n} = \frac{-2ax-b}{(n-1)\left(b^2-4ac\right)\left(ax^2+bx+c\right)^{n-1}} - \frac{2(2n-3)a}{(n-1)\left(b^2-4ac\right)} \int \frac{dx}{\left(ax^2+bx+c\right)^{n-1}}, n \neq 1$

- $\int \frac{dx}{\left(x^2+a^2\right)^n} = \frac{x}{2(n-1)a^2\left(x^2+a^2\right)^{n-1}} + \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{\left(x^2+a^2\right)^{n-1}}, n \neq 1$

- $\int \frac{dx}{\left(x^2-a^2\right)^n} = \frac{x}{2(n-1)a^2\left(x^2a^2\right)^{n-1}} - \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{\left(x^2-a^2\right)^{n-1}}, n \neq 1$

8. Derived substitutions:

A. Algebraic Twins

- $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx,$

- $\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$

- $\int \frac{2x^2}{\left(x^4+1+kx^2\right)}dx \cdot \int \frac{2}{\left(x^4+1+kx^2\right)}dx$

B. Trigonometric twins

- $\int \sqrt{\tan x}\,dx, \int \sqrt{\cot x}\,dx,$

- $\int \frac{1}{\left(\sin^4 x+\cos^4 x\right)}dx, \int \frac{1}{\left(\sin^6 x+\cos^6 x\right)}dx, \int \frac{\pm\sin x \pm \cos x}{a+b\sin x\cos x}dx.$